Y. Nakamura, Department of Mechanical Engineering, Kagoshima University

Fundamentals for Mechanical Design of Internally Pressurized Cylinders

In a cylindrical coordinate system in which the r and θ axes are related by $r=\sqrt{x^2+y^2}$ and $\theta=\tan y/x$ with the orthogonal xy axes, the strains in $r\theta$ plane are given by

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)$$

where u_r and u_θ are the displacements in radial and tangential directions, respectively.

In particular, when deformation at any point in the cylinder takes place symmetrically around the origin and only the radial displacement is induced, the above relations are simplified as follows.

$$\varepsilon_{rr}=\frac{du_r}{dr}, \quad \varepsilon_{\theta\theta}=\frac{u_r}{r}, \quad \varepsilon_{r\theta}=0$$

In this case, the compatibility condition of strains , which is required for the continuity of displacement, is given by

$$\frac{d\varepsilon_{\theta\theta}}{dr}=\frac{1}{r}\frac{du_r}{dr}-\frac{u_r}{r^2}=\frac{\varepsilon_{rr}-\varepsilon_{\theta\theta}}{r}$$

As for the elastic deformation of isotropic materials, according to the Hooke's law, strains are related with stresses as follows.

$$\varepsilon_{rr}=\frac{\sigma_{rr}-\nu(\sigma_{\theta\theta}+\sigma_{zz})}{E}, \quad \varepsilon_{\theta\theta}=\frac{\sigma_{\theta\theta}-\nu(\sigma_{rr}+\sigma_{zz})}{E}, \quad \varepsilon_{zz}=\frac{\sigma_{zz}-\nu(\sigma_{rr}+\sigma_{\theta\theta})}{E}$$

$$\varepsilon_{r\theta}=\varepsilon_{\theta r}=\varepsilon_{\theta z}=\varepsilon_{z\theta}=0, \quad \varepsilon_{rz}=\varepsilon_{zr}=\frac{\sigma_{rz}}{2G}$$

where E, G and ν are the Young's modulus, shear modulus and Poisson's ratio, respectively. The shear modulus can be related with the Young's modulus and Poisson's ratio as follows.

$$G=\frac{E}{2(1+\nu)}$$

When the thickness of the cylinder is small, we can ignore the stress components related with z axis, *i.e.*, $\sigma_{zz}=0$ and $\sigma_{rz}=0$. This stress state is called "plane stress", and the strain components are given by

$$\varepsilon_{rr}=\frac{\sigma_{rr}-\nu\sigma_{\theta\theta}}{E}, \quad \varepsilon_{\theta\theta}=\frac{\sigma_{\theta\theta}-\nu\sigma_{rr}}{E}, \quad \varepsilon_{zz}=-\frac{\nu(\sigma_{rr}+\sigma_{\theta\theta})}{E}$$

$$\varepsilon_{r\theta}=\varepsilon_{\theta r}=\varepsilon_{\theta z}=\varepsilon_{z\theta}=\varepsilon_{rz}=\varepsilon_{zr}=0$$

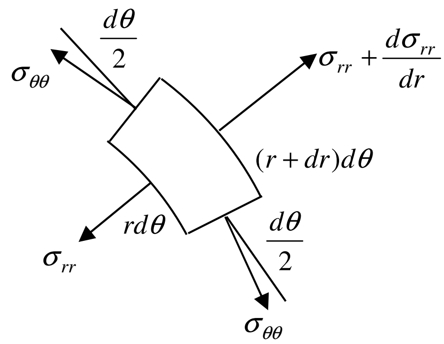
On the other hand, the thickness of the cylinder is large, we can ignore the strain components related with z axis, *i.e.*, $\varepsilon_{zz}=\varepsilon_{rz}=0$. This strain state is called "plane strain", and we obtain

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})}{E} = 0 \quad \text{and therefore} \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

This relation leads to

$$\varepsilon_{rr} = \frac{(1-\nu^2)\sigma_{rr} - \nu(1+\nu)\sigma_{\theta\theta}}{E}, \quad \varepsilon_{\theta\theta} = \frac{(1-\nu^2)\sigma_{\theta\theta} - \nu(1+\nu)\sigma_{rr}}{E}$$

The equilibrium of forces acting on an infinitesimally small element shown by the right figure leads to



$$\left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r}\right)(r+dr)d\theta = \sigma_{rr} r d\theta + 2\sigma_{\theta\theta} dr \frac{d\theta}{2}$$

Neglecting the second-order small amounts, we obtain

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

This equation indicates the equilibrium condition of stresses. Now we set σ_{rr} and $\sigma_{\theta\theta}$ to be

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr}, \quad \sigma_{\theta\theta} = \frac{d^2\phi}{dr^2}$$

By inserting these equations into the equilibrium condition of stresses, we can see

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right) + \frac{1}{r^2}\frac{d\phi}{dr} - \frac{1}{r}\frac{d^2\phi}{dr^2} = -\frac{1}{r^2}\frac{d\phi}{dr} + \frac{1}{r}\frac{d^2\phi}{dr^2} + \frac{1}{r^2}\frac{d\phi}{dr} - \frac{1}{r}\frac{d^2\phi}{dr^2} = 0$$

Thus the stress function ϕ satisfies the equilibrium of stresses.

In case of the plane strain, the compatibility condition of strains given by

$$\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\varepsilon_{rr} - \varepsilon_{\theta\theta}}{r} = 0$$

can be transformed into the following equation.

$$\begin{aligned}
&\frac{d\varepsilon_{\theta\theta}}{dr} - \frac{\varepsilon_{rr} - \varepsilon_{\theta\theta}}{r} \\
&= \frac{(1-\nu^2)d\sigma_{\theta\theta}/dr - \nu(1+\nu)d\sigma_{rr}/dr}{E} - \frac{(1-\nu^2)\sigma_{rr} - \nu(1+\nu)\sigma_{\theta\theta} - (1-\nu^2)\sigma_{\theta\theta} + \nu(1+\nu)\sigma_{rr}}{Er} \\
&= \frac{1}{E}\left\{(1-\nu^2)\frac{d\sigma_{\theta\theta}}{dr} - \nu(1+\nu)\frac{d\sigma_{rr}}{dr} - (1+\nu)\frac{\sigma_{rr}}{r} + (1+\nu)\frac{\sigma_{\theta\theta}}{r}\right\} \\
&= \frac{1+\nu}{E}\left\{(1-\nu)\frac{d^3\phi}{dr^3} - \nu\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right) - \frac{1}{r^2}\frac{d\phi}{dr} + \frac{1}{r}\frac{d^2\phi}{dr^2}\right\} \\
&= \frac{1+\nu}{E}\left\{(1-\nu)\frac{d^3\phi}{dr^3} + \nu\frac{1}{r^2}\frac{d\phi}{dr} - \nu\frac{1}{r}\frac{d^2\phi}{dr^2} - \frac{1}{r^2}\frac{d\phi}{dr} + \frac{1}{r}\frac{d^2\phi}{dr^2}\right\} \\
&= \frac{(1+\nu)(1-\nu)}{E}\left(\frac{d^3\phi}{dr^3} + \frac{1}{r}\frac{d^2\phi}{dr^2} - \frac{1}{r^2}\frac{d\phi}{dr}\right) \\
&= 0
\end{aligned}$$

As a result we obtain

$$\frac{d^3\phi}{dr^3}+\frac{1}{r}\frac{d^2\phi}{dr^2}-\frac{1}{r^2}\frac{d\phi}{dr}=0$$

Now we can find the following relation.

$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right\}=\frac{d}{dr}\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{d^3\phi}{dr^3}+\frac{1}{r}\frac{d^2\phi}{dr^2}-\frac{1}{r^2}\frac{d\phi}{dr}$$

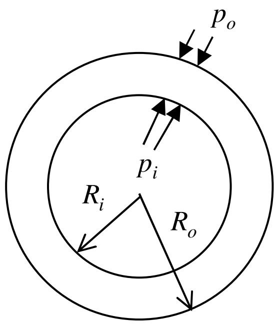
Therefore

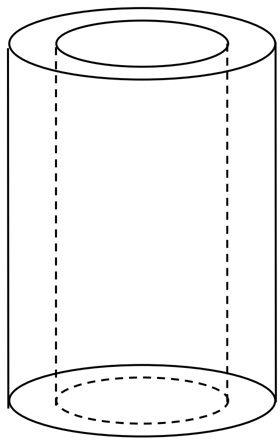
$$\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)\right\}=0\,,\quad \frac{d}{dr}\left(r\frac{d\phi}{dr}\right)=2Ar\,,\quad \frac{d\phi}{dr}=Ar+\frac{B}{r}$$

$$\phi(r)=\frac{Ar^2}{2}+B\ln r+C$$

where *A*, *B* and *C* are constants which can be determined by using boundary conditions. Accordingly the stresses are given by

$$\sigma_{rr}=\frac{1}{r}\frac{d\phi}{dr}=A+\frac{B}{r^2}\,,\quad \sigma_{\theta\theta}=\frac{d^2\phi}{dr^2}(r)=A-\frac{B}{r^2}\,,\quad \sigma_{zz}=\nu(\sigma_{rr}+\sigma_{\theta\theta})=2\nu A$$





Now we set the pressures at the inner and outer boundaries of the cylinder to be p_i and p_o. and also set the inner and outer radii to be R_i and R_o, respectively. By using these boundary conditions, we obtain

$$\sigma_{rr}(R_i)=A+\frac{B}{R_i^{\,2}}=-p_i\,,\quad \sigma_{rr}(R_o)=A+\frac{B}{R_o^{\,2}}=-p_o$$

and these relations lead to

$$A=\frac{p_iR_i^{\,2}-p_oR_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}\,,\quad B=-\frac{(p_i-p_o)R_i^{\,2}R_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}$$

Accordingly the stresses become

$$\sigma_{rr}=\frac{p_iR_i^{\,2}-p_oR_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}-\frac{(p_i-p_o)R_i^{\,2}R_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}\frac{1}{r^2}$$

$$\sigma_{\theta\theta}=\frac{p_iR_i^{\,2}-p_oR_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}+\frac{(p_i-p_o)R_i^{\,2}R_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}\frac{1}{r^2}$$

$$\sigma_{zz}=2\nu\frac{p_iR_i^{\,2}-p_oR_o^{\,2}}{R_o^{\,2}-R_i^{\,2}}$$

When the internal pressure p_i is much larger than the outer pressure p_o, the above equations can be approximated as

$$\sigma_{rr}\approx\frac{p_iR_i^{\,2}}{R_o^{\,2}-R_i^{\,2}}\left(1-\frac{R_o^{\,2}}{r^2}\right),\quad \sigma_{\theta\theta}\approx\frac{p_iR_i^{\,2}}{R_o^{\,2}-R_i^{\,2}}\left(1+\frac{R_o^{\,2}}{r^2}\right),\quad \sigma_{zz}\approx 2\nu\frac{p_iR_i^{\,2}}{R_o^{\,2}-R_i^{\,2}}$$

According to plasticity theory, the Tresca's yield criterion predicts that the material deforms plastically when the following equation is satisfied.

$$\sigma_1 - \sigma_3 \geq \sigma_Y ,$$

where σ_1 and σ_3 are the principal stresses of which values are maximum and minimum, respectively, and σ_Y is the yield strength obtained by uniaxial tests. We can see from the equations which are derived for elastic deformation that $\sigma_1 = \sigma_{\theta\theta}$ and $\sigma_3 = \sigma_{rr}$. Thus we obtain the following yielding criterion.

$$\sigma_{\theta\theta} - \sigma_{rr} \approx \frac{2p_i R_i^2}{R_o^2 - R_i^2}\frac{R_o^2}{r^2} = \sigma_Y$$

The value of deviatoric stress given by $\sigma_{\theta\theta} - \sigma_{rr}$ becomes largest at the inner boundary. Hence the condition under which no plastic deformation occurs in the cylinder is given by

$$\frac{2p_i R_o^2}{R_o^2 - R_i^2} < \sigma_Y \text{, and therefore } R_o > \sqrt{\frac{\sigma_Y}{\sigma_Y - 2p_i}} R_i$$

For example, if $\sigma_Y = 200\,\text{MPa}$ and $p_i = 50\,\text{MPa}$, then the ratio of outer and inner radii becomes

$$\frac{R_o}{R_i} > \sqrt{2} = 1.41$$

It should be noticed that this ratio is minimum. The practical design requires some safety factor which ensures that real stresses are well below the stresses calculated above. From this reason, by using the safety factor *S*, we obtain

$$\frac{2p_i R_o^2}{R_o^2 - R_i^2} < \frac{\sigma_Y}{S} \text{ and therefore } R_o > \sqrt{\frac{\sigma_Y}{\sigma_Y - 2Sp_i}} R_i$$

It is also noted that if the radii and yield strength of the cylinder is known, the pressure should satisfy

$$p_i < \left(1 - \frac{R_i^2}{R_o^2}\right)\frac{\sigma_Y}{2S}$$

in order to prevent the occurrence of plastic deformation in the cylinder.

Next we consider the case in which the internal pressure is larger than that limited by the yield criterion. In this case plastic deformation takes place from the inner surface and propagates in the cylinder as the pressure is increased. If the material does not work harden after yielding, the largest deviation stress in the plastic region maintains the relation given by $\sigma_{\theta\theta} - \sigma_{rr} = \sigma_Y$. Accordingly the equilibrium of stresses in the plastic region is given by

$$\frac{d\sigma_{rr}}{dr} - \frac{\sigma_Y}{r} = 0 \quad \text{for} \quad R_i \leq r \leq R_p$$

where R_p is the radius of the plastic region. This equation leads to

$$\sigma_{rr} = \sigma_Y \ln r + D$$

and by using the boundary condition that $\sigma_{rr}(R_i) = -p_i$, the constant *D* is found to be

$$D = -\sigma_Y \ln R_i - p_i$$

As a result the stresses in the plastic region are given by

$$\sigma_{rr} = \sigma_Y \ln\frac{r}{R_i} - p_i , \quad \sigma_{\theta\theta} = \sigma_Y \ln\frac{r}{R_i} - p_i + \sigma_Y$$

On the other hand, the inner boundary of the surrounding elastic region must satisfy the following relation.

$$\sigma_{\theta\theta}(R_p) - \sigma_{rr}(R_p) \approx \frac{2qR_i^2}{R_o^2 - R_i^2}\frac{R_o^2}{R_p^2} = \sigma_Y$$

where q is the pressure at the boundary of $r = R_p$, which is given by

$$q = -\sigma_{rr}(R_p) = p_i - \sigma_Y \ln\frac{R_p}{R_i}$$

Therefore we obtain

$$p_i = \left\{\left(1 - \frac{R_i^2}{R_o^2}\right)\frac{R_p^2}{R_i^2} + \ln\frac{R_p}{R_i}\right\}\frac{\sigma_Y}{2}$$

It is clear from this equation that the yielding from the inner surface to the outer surface in the cylinder takes place when

$$p_i \geq \left(\frac{R_o^2}{R_i^2} + \ln\frac{R_o}{R_i} - 1\right)\frac{\sigma_Y}{2}$$

In this case we cannot prevent the plastic deformation from spreading through the thickness of cylinder and the cylinder is expected to break.